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Spectral properties of non-conservative multichannel SUSY partners of the zero potential

Andrey M Pupasov^{1,2,3}, Boris F Samsonov¹ and Jean-Marc Sparenberg²

¹ Physics Department, Tomsk State University, 36 Lenin Avenue, 634050 Tomsk, Russia

² Physique Quantique, CP 229, Université Libre de Bruxelles, B-1050 Bruxelles, Belgium

E-mail: pupasov@phys.tsu.ru, samsonov@phys.tsu.ru and jmspar@ulb.ac.be

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Abstract

Spectral properties of a coupled $N \times N$ potential model obtained with the help of a single non-conservative supersymmetric (SUSY) transformation starting from a system of N radial Schrödinger equations with the zero potential and finite threshold differences between the channels are studied. The structure of the system of polynomial equations which determine the zeros of the Jost-matrix determinant is analyzed. In particular, we show that the Jost-matrix determinant has $N2^{N-1}$ zeros which may all correspond to virtual states. The number of bound states satisfies $0 \leq n_b \leq N$. The maximal number of resonances is $n_r = (N-1)2^{N-2}$. A perturbation technique for a small coupling approximation is developed. A detailed study of the inverse spectral problem is given for the 2×2 case.

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1. Introduction

Almost all low-energy collisions of microparticles with an internal structure (i.e., atom–atom, nucleus–nucleus, etc) include inelastic processes such as excitations of internal degrees of freedom of colliding particles or processes with rearrangements of their constituent parts. These processes can be described by a matrix (more precisely multichannel) Schrödinger equation with a local matrix potential [1, 2]. One may be interested in both direct and inverse scattering problems for this equation. The method of SUSY transformations is known as a powerful tool for solving both types of problems for a single-channel Schrödinger equation [3]. Nowadays, the first attempt to generalize the method for a coupled-channel Schrödinger equation with different thresholds is given in [4, 5]. This attempt is based on a non-conservative SUSY transformation (contrary to [6, 7]), i.e. a SUSY transformation that does not preserve a

³ Boursier de l'ULB.

boundary behavior of solutions. The main advantage of such transformations is a possibility to obtain multichannel potentials with a non-trivial coupling starting from the zero potential.

The present work is aimed at the investigation of spectral properties of these SUSY potentials. Our approach is based on an analysis of the Jost matrix. In the non-relativistic scattering theory, the Jost matrix plays a fundamental role similar to the scattering matrix. The zeros of the Jost-matrix determinant define positions of the bound/virtual states and resonances [1, 2]. Therefore, studying the zeros of the Jost-matrix determinant allows one to analyze the spectrum of the model. A closed analytical expression of the Jost matrix, as well as potential, resulting from a non-conservative SUSY transformation of the zero potential is obtained in [4]. The analysis of spectral properties for such potentials has not been presented up to now despite the fact that the Jost matrix is well known [8]. This may be explained by the fact that the spectrum of the potential after a non-conservative SUSY transformation changes essentially and to find these changes one has to find all the zeros of the Jost-matrix determinant. More precisely, no spectral point of the initial Hamiltonian belongs to the spectrum of the transformed Hamiltonian. As a result, a supersymmetry algebra, which is always present in the case of conservative SUSY transformations, cannot actually be constructed here and the word ‘SUSY transformation’ is only a formal heritage from the previous conservative case [6, 7].

The principal point of this paper is to show that the qualitative behavior of the spectrum of (non-conservative) SUSY partners of the vanishing multichannel potential with threshold differences may be studied for an arbitrary number of channels, N . We think this is a very strong result, since even for the case $N = 2$ the full analysis of the spectrum is a very complicated problem [8, 10, 11]. The main reason for this is an extremely rapid growth of the order of an algebraic equation defining the spectrum with the growth of the number of channels.

The paper is organized as follows. We start with preliminaries, where we give basic definitions and equations. Section 3 is devoted to the analysis of the number of bound states resulting from a non-conservative SUSY transformation of the zero potential as a function of the parameters defining the transformation. This analysis is based on the study of the properties of the eigenvalues of the Jost matrix. Following similar lines we analyze the possible number of virtual states in section 4. Once the bound and virtual states are analyzed we can formulate the conditions under which resonances may appear; this is made in section 5. The behavior of the Jost-matrix determinant zeros is studied in section 6 in the approximation of a weak coupling between channels. In section 7 we deal with the particular two-channel case. In this case, we express parameters of the potential in terms of the zeros of the Jost-matrix determinant, i.e. solve an inverse spectral problem. The main results are summarized in the conclusion.

2. Preliminaries

Let us first summarize the notations used below for coupled-channel scattering theory [1, 2, 9]. We consider a system of coupled radial Schrödinger equations for the s -waves that in reduced units read

$$H\psi(k, r) = K^2\psi(k, r), \quad r \in (0, \infty), \quad (1)$$

with

$$H = -\mathbf{1} \frac{d^2}{dr^2} + V(r), \quad (2)$$

where r is the radial coordinate, $V(r)$ is an $N \times N$ real symmetric matrix, $\mathbf{1}$ is the unit matrix and ψ may be either a matrix-valued or a vector-valued solution. By k we denote a point in

the space \mathbb{C}^N , $k = \{k_1, \dots, k_N\}$, $k_i \in \mathbb{C}$. A diagonal matrix with non-vanishing entries k_i is written as $K = \text{diag}(k) = \text{diag}(k_1, \dots, k_N)$. The complex wave numbers k_i are related to the center-of-mass energy E and the channel thresholds $\Delta_1, \dots, \Delta_N$, which are supposed to be different from each other, $\Delta_{i(\neq j)} \neq \Delta_j$, by

$$k_j^2 = E - \Delta_j, \quad \Delta_1 = 0. \quad (3)$$

We assume here that $\Delta_1 = 0$ and the different channels have equal reduced masses, a case to which the general situation can always be formally reduced [2].

Let us recall basic definitions from SUSY quantum mechanics [3–7]. It is known that the solutions of the initial Schrödinger equation (1) may be mapped into the solutions of the transformed equation with the help of the differential-matrix operator

$$\tilde{\psi}(k, r) = L\psi(k, r) = \left[-\mathbf{1} \frac{d}{dr} + U(r) \right] \psi(k, r). \quad (4)$$

The transformed Schrödinger equation has form (1) with a new potential

$$\tilde{V}(r) = V(r) - 2U'(r). \quad (5)$$

The matrix U is called superpotential

$$U(r) = \eta'(r)\eta^{-1}(r), \quad (6)$$

and expressed in terms of a matrix solution η of the initial Schrödinger equation

$$H\eta(r) = -\mathcal{K}^2\eta(r), \quad (7)$$

where $\mathcal{K} = \text{diag}(\kappa) = \text{diag}(\kappa_1, \dots, \kappa_N)$ is a diagonal matrix called the factorization wave number, which corresponds to an energy \mathcal{E} lying below all thresholds, called the factorization energy. The entries of \mathcal{K} , thus, satisfy $\mathcal{E} = -\kappa_i^2 + \Delta_i$; by convention, we choose them positive: $\kappa_i > 0$. The solution η is called the factorization solution.

In the case of the zero potential $V = 0$, η contains only exponentials

$$\eta(r) = \cosh(\mathcal{K}r) + \mathcal{K}^{-1} \sinh(\mathcal{K}r)U_0. \quad (8)$$

The symmetric matrix U_0 is the superpotential at $r = 0$, which can be chosen arbitrarily. It is convenient to introduce special notations α_j for the diagonal and β_{jl} for the off-diagonal entries of U_0 .

The Jost matrix of a (non-conservative) SUSY partner of the N -channel zero potential reads [4]

$$F(k) = (\mathcal{K} - iK)^{-1}(U_0 - iK), \quad (9)$$

which is also the Jost matrix obtained in [8].

The necessary and sufficient condition on the parameters (factorization energy \mathcal{E} and superpotential at the origin U_0) to get a potential without singularity at finite distances is obtained in [10, 11]. This condition is the positive definiteness of the matrix $\mathcal{K} + U_0$:

$$\mathcal{K} + U_0 > 0, \quad (10)$$

which puts some upper limit on the factorization energy \mathcal{E} at a fixed U_0 .

The zeros of the Jost-matrix determinant define positions of the bound/virtual states and the resonances. Thus, to find these positions we have to solve the following equation:

$$\det F(k) = 0, \quad (11)$$

taking into account the threshold conditions (3). According to (9), the roots of equation (11) are defined by the roots of

$$\det B(k) = 0, \quad \kappa_j - ik_j \neq 0, \quad j = 1, \dots, N, \quad (12)$$

where

$$B(k) = U_0 - iK. \tag{13}$$

In what follows, we concentrate on the analysis of the zeros of $\det B$ only keeping in mind that some of them may be canceled in $\det F$ if $k_j = -i\kappa_j$. Our starting point is thus a system of algebraic equations (12) and (3) which reads, with certain coefficients a_i^j ,

$$(-i)^N k_1 k_2 \cdots k_N + \sum_{j=1}^N a_{N-1}^j \prod_{l=1, l \neq j}^N k_l + \cdots + \sum_{j=1}^N a_1^j k_j + a_0 = 0, \tag{14}$$

$$k_j^2 - k_1^2 + \Delta_j = 0. \tag{15}$$

First, we show that system (14), (15) can be reduced to an algebraic equation of the $N2^{N-1}$ degree with respect to one momentum, say k_1 , only. Indeed, any momentum enters equation (14) only linearly. Therefore, it can be rewritten in the form

$$k_N P_1(k_1, \dots, k_{N-1}) = Q_1(k_1, \dots, k_{N-1}), \tag{16}$$

where $P_1(k_1, \dots, k_{N-1})$ and $Q_1(k_1, \dots, k_{N-1})$ are the polynomials of first degree in each of the variables k_1, \dots, k_{N-1} . It is important to note that given all momenta k_1, \dots, k_{N-1} this equation defines k_N in a unique way if P_1 does not vanish. On the other hand, we can square the left- and right-hand sides of (16) thus obtaining an equation where k_N enters only in the second degree and polynomials P_1^2 and Q_1^2 are polynomials of the second degree with respect to their variables. But in the equation thus obtained using threshold condition (15), we can replace all second powers of the variables $k_j, j = 2, \dots, N$ by $k_1^2 - \Delta_j$, which makes disappear both the variable k_N and the second power of $k_j, j = 2, \dots, N - 1$, from the resulting equation and raises the power of k_1 till $2N$. We thus see that after these manipulations the variable k_{N-1} enters in the resulting equation only in the first degree and the equation can be rewritten in form (16)

$$k_{N-1} P_2(k_1, \dots, k_{N-2}) = Q_2(k_1, \dots, k_{N-2}), \tag{17}$$

where $P_2(k_1, \dots, k_{N-2})$ and $Q_2(k_1, \dots, k_{N-2})$ are polynomials of the first degree in each of the variables k_2, \dots, k_{N-2} . From (17), given k_1, \dots, k_{N-2} , not a zero of P_{N-2} , we obtain k_{N-1} in a unique way. We note that system (17), (16) and (15), where from (15) the last equation $k_N^2 - k_1^2 + \Delta_N = 0$ should be excluded, is equivalent to the original system (14), (15).

It is clear that we can repeat the above process $N - 3$ times more to get an equation

$$k_2 P_{N-1}(k_1) = Q_{N-1}(k_1), \tag{18}$$

and finally

$$P_N(k_1) = 0 \tag{19}$$

with P_N of the order $N2^{N-1}$. Note that the subscript in P_k and Q_k indicates nothing but the step in this procedure. It is evident that any k_1 which (together with k_2, \dots, k_N) solves system (14), (15) is a root of (19). The converse is also true. Indeed, given a root k_1 of (19), but not a root of P_{N-1} , we find from (18) a unique k_2 . Once we know k_1 and k_2 , we find k_3 from the equation previous to (18) and so on till k_N which is found from (16). It is also clear that in this way we can get $N2^{N-1}$ number of sets k_1, \dots, k_N (some of them may coincide) each of which solves system (14), (15) so that the same number $N2^{N-1}$ is the number of possible solutions of this system and system (19), (18), \dots , (16) is equivalent to the initial system (14), (15).

3. The number of bound states

In the following, except for sections 6 and 7, we will consider all quantities as the functions of the momentum k_1 . Other momenta are expressed in terms of k_1 from the threshold conditions (3). Since in this section we are interested in the number of bound states, we will consider only the negative energy semi-axis $E \in (-\infty, 0)$. It happens to be useful to change the variables k_j in favor of \bar{k}_j as $k_j = i\bar{k}_j$ and rewrite the threshold conditions (15) accordingly

$$\bar{k}_j = \sqrt{\bar{k}_1^2 + \Delta_j}, \tag{20}$$

where we have chosen only the positive value of the square root since in this section we analyze only the point spectrum of H , which restricts all momenta k_j to be purely imaginary with a positive imaginary part so that $\bar{k}_j = |k_j|$.

From (9) it is clear that all the zeros of $\det F$ are at the same time the zeros of the determinant of the matrix B (13) and vice versa. This follows from (12) and the positive definiteness of the matrix $\mathcal{K} - iK$ in the momenta region we consider so that neither of the roots of $\det B$ solves the equation $\det(\mathcal{K} - iK) = 0$.

Since $\det B = \prod_{j=1}^N \lambda_j$, where λ_j are the eigenvalues of B ,

$$B(\bar{k}_1)x_j(\bar{k}_1) = \lambda_j(\bar{k}_1)x_j(\bar{k}_1), \tag{21}$$

the equation $\det B(\bar{k}_1) = 0$ is equivalent to $\lambda_j(\bar{k}_1) = 0, j = 1, \dots, N$. The matrix B is symmetric with real entries in the momenta region we consider, $B = U_0 + \bar{K} = B^T$, which implies the reality of both $\lambda_j(\bar{k}_1)$ and $x_j(\bar{k}_1)$. Here we introduced a diagonal matrix $\bar{K} = |K| = \text{diag}(\bar{k}_1, \dots, \bar{k}_N)$.

Another property of $\lambda_j(\bar{k}_1)$ important for the analysis is their monotony as functions of \bar{k}_1 that we prove below.

For a fixed \bar{K} let us consider a deviation of $\lambda_j(\bar{k}_1)$ for a small increment of the argument \bar{k}_1 , i.e. $\lambda_j(\bar{k}_1 + \delta\bar{k}_1) = \lambda_j(\bar{k}_1) + \delta\lambda_j(\bar{k}_1)$ assuming $\delta\bar{K} = \text{diag}(\delta\bar{k}_1, \dots, \delta\bar{k}_N)$ real, positive definite (since $\delta\bar{k}_j > 0, \forall j$) and infinitesimal. From (21) one gets

$$B(\bar{k}_1 + \delta\bar{k}_1)x_j(\bar{k}_1 + \delta\bar{k}_1) = \lambda_j(\bar{k}_1 + \delta\bar{k}_1)x_j(\bar{k}_1 + \delta\bar{k}_1). \tag{22}$$

Here according to (13) $B(\bar{k}_1 + \delta\bar{k}_1) = U_0 + \bar{K} + \delta\bar{K}$, and the increment of $B(\bar{k}_1)$ is just $\delta B = \delta\bar{K}$ which plays the role of a small perturbation of $B(\bar{k}_1)$. Therefore, we may calculate the shifting of the eigenvalues produced by such a perturbation using a (Rayleigh–Schrödinger) perturbation theory. Thus, for a non-degenerate eigenvalue λ_j , the first-order correction reads

$$\delta\lambda_j = \langle x_j | \delta B | x_j \rangle > 0, \tag{23}$$

where the inequality follows from the positive definiteness of $\delta B = \delta\bar{K}$, which in turn implies monotony of the eigenvalues as the functions of the momenta \bar{k}_1 . For a degenerate eigenvalue, corrections are obtained by diagonalizing the same perturbation operator δB restricted to a linear span of unperturbed eigenvectors corresponding to a given eigenvalue, which still leads to positive corrections because of positive definiteness of δB .

From here it follows that any eigenvalue $\lambda_j(\bar{k}_1)$ may vanish, i.e. change its sign, only once. Moreover, $\lambda_j \rightarrow \bar{k}_j > 0$ as $\bar{k}_1 \rightarrow \infty$. Hence, the number of negative eigenvalues of B at $\bar{k}_1 = 0$, i.e. at the energy of the lowest threshold, is just the number of bound states. Thus, to count the number of bound states, n_b , one has to consider the eigenvalues $\lambda_j(\bar{k}_1), j = 1, \dots, N$, of the matrix $B(\bar{k}_1)$ at $\bar{k}_1 = 0$,

$$B(0) \equiv U_0 - i \text{diag}(i\sqrt{\Delta_j}) = U_0 + \text{diag}(\sqrt{\Delta_j}), \tag{24}$$

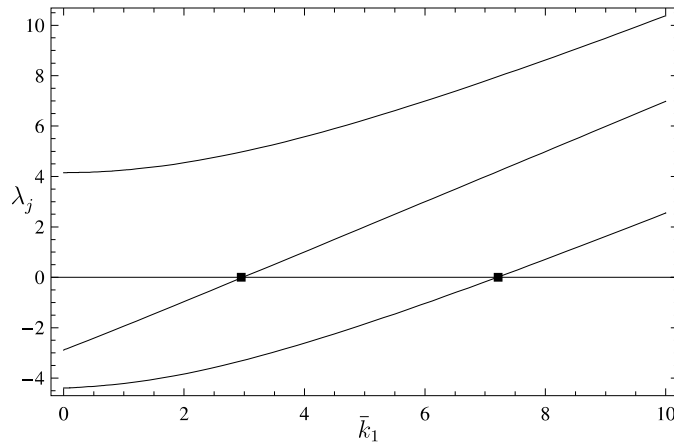


Figure 1. Typical behavior of the B -matrix eigenvalues, $N = 3$. The case of two bound states with energies $E_1 = -51.8611$ and $E_2 = -8.8852$ is presented. The black squares show positions of these bound states. The corresponding parameters are $\alpha_1 = -3, \alpha_2 = -8, \alpha_3 = -1, \beta_{12} = 0.5, \beta_{13} = 0.4, \beta_{23} = 1, \Delta_2 = 15$ and $\Delta_3 = 25$.

so that

$$n_b = \frac{1}{2}(N - \Lambda), \quad \Lambda = \sum_{j=1}^N \Lambda_j, \quad \Lambda_j = \frac{\lambda_j(0)}{|\lambda_j(0)|}. \quad (25)$$

To clarify this formula, we note that in the absence of bound states all $\Lambda_j = 1, \Lambda = N$ so that $n_b = 0$. Every bound state is responsible for the change of the sign of only one eigenvalue from positive to negative thus raising $-\Lambda$ by two units, i.e. $-\Lambda \rightarrow -\Lambda + 2$ with $n_b \rightarrow n_b + 1$. This justifies the factor $1/2$ in (25).

Summarizing, we see that the number of bound states is bounded by $0 \leq n_b \leq N$. Figure 1 shows the eigenvalues of the matrix B as the functions of \bar{k}_1 for the case $N = 3$. Two eigenvalues cross the axis which corresponds to the case of $n_b = 2$. The last comment in this section is devoted to equation (10). Now it can be seen that the factorization energy should be chosen lower than the ground-state energy for the transformed potential, $\mathcal{E} < E_g$, if any.

4. The number of virtual states

According to the definition of a virtual state [1, 2], in this section we will need to consider the channel wave numbers k_j lying both in the positive and the negative imaginary semi-axes of the corresponding momenta complex planes and consider the full imaginary axis for k_1 , i.e. $\bar{k}_1 \in (-\infty, \infty)$. The other momenta, k_2, \dots, k_N , belong to either the positive or the negative parts of their imaginary axes in agreement with the threshold conditions

$$\bar{k}_j = \sigma_j \sqrt{\bar{k}_1^2 + \Delta_j}, \quad \sigma_j = \pm, \quad j = 2, \dots, N. \quad (26)$$

Since in (26) all combinations of signs are now possible it is convenient to introduce special notations for these combinations. Denote $\sigma = (+, \pm, \dots, \pm)$ a string of N signs with σ_j being its j th entry, which corresponds to the sign in (26) for the j th momentum for $j > 1$. The first symbol ‘+’ in σ indicates that all the momenta \bar{k}_j are expressed in terms of \bar{k}_1 . Let $n_+(\sigma) + 1$ and $n_-(\sigma)$ be the numbers of ‘+’ and ‘-’ signs in this string. We note the following

evident combinatoric properties of $n_-(\sigma)$ and $n_+(\sigma)$. First, $n_+(\sigma) + n_-(\sigma) + 1 = N$ which implies

$$\sum_{\sigma} [n_+(\sigma) + n_-(\sigma) + 1] = N2^{N-1}. \tag{27}$$

Here and in what follows the summation over σ includes all 2^{N-1} possible sign combinations. Next, a symmetry between ‘+’ and ‘-’ leads to the following relation:

$$\sum_{\sigma} n_-(\sigma) = \sum_{\sigma} n_+(\sigma) = (N - 1)2^{N-2}. \tag{28}$$

According to (13) every sign combination leads to its own B matrix defined by the corresponding K matrix so that both K and B should carry an additional information about this combination. Therefore,

$$B^{\sigma} = U_0 + \bar{K}^{\sigma}, \quad \bar{K}^{\sigma} = \text{diag}(\bar{k}_1, \sigma_2 \bar{k}_2, \dots, \sigma_N \bar{k}_N) \tag{29}$$

and we denote $\lambda_j^{\sigma}(\bar{k}_1)$, $j = 1, \dots, N$ the eigenvalues of B^{σ} .

In order to find the zeros of the Jost-matrix determinant corresponding to the virtual states, we should find the purely real solutions of the equations $\lambda_j^{\sigma}(\bar{k}_1) = 0$, $j = 1, \dots, N$, for all 2^{N-1} matrices B^{σ} . Although \bar{k}_j ’s are real, but bearing in mind our replacement $k_j = i\bar{k}_j$, throughout the text we call these zeros purely imaginary. Finally, we note that since the matrix $\mathcal{K} - iK$ in (9) is not positive definite for an arbitrary σ anymore, in some particular cases some of the zeros of B may be canceled by the zeros of $\det(\mathcal{K} - iK)$ and will not correspond to virtual states. Nevertheless, omitting these particular cases, we will concentrate on an analysis of the zeros of $\det B$ only.

The eigenvalues $\lambda_j^{\sigma}(\bar{k}_1)$ are monotonous functions of \bar{k}_1 in two cases only: (i) $\sigma = (+, +, \dots, +)$ and $\bar{k}_1 > 0$; (ii) $\sigma = (+, -, \dots, -)$ and $\bar{k}_1 < 0$. In general, an eigenvalue $\lambda_j^{\sigma}(\bar{k}_1)$ may have minima/maxima for $\bar{k}_1 \leq 0$ which may lead to two or even more roots in the equation $\lambda_j^{\sigma}(\bar{k}_1) = 0$. We illustrate this behavior for $N = 3$ in figure 2. The monotonous lines in the right/left part of figure 2(a)/(c) correspond to the case (i)/(ii). The position of the zeros of the eigenvalues is shown by stars, squares and circles. It is clearly seen that the total number of roots of all equations $\lambda_j^{\sigma}(\bar{k}_1) = 0$ is $(N2^{N-1})|_{N=3} = 12$ which all correspond to virtual states.

A change of parameters may result in shifting the position of the virtual states only without changing the number of zeros (i.e. virtual states). For instance, in the simplest case we may shift all diagonal entries of U_0 by a number λ_0 , $U_0 \rightarrow U_0 + \lambda_0 I$, thus shifting all eigenvalues of B by the same number, $\lambda_j^{\sigma}(\bar{k}_1) \rightarrow \lambda_j^{\sigma}(\bar{k}_1) + \lambda_0$.

Let us consider a specific eigenvalue defined by a string σ_0 , with a local maximum at $\bar{k}_1 = \bar{k}_{1,\text{max}}$, $\lambda_j^{\sigma_0}(\bar{k}_{1,\text{max}}) = \lambda_{j,\text{max}}$. One can always shift all the eigenvalues by the value $\lambda_{j,\text{max}}$ such that the curve $\lambda_j^{\sigma_0}(\bar{k}_1)$ touches the \bar{k}_1 axis at the point $\bar{k}_1 = \bar{k}_{1,\text{max}}$, meaning that $\bar{k}_{1,\text{max}}$ not only becomes a root of the equation $\lambda_j^{\sigma_0}(\bar{k}_1) = 0$ but this root is a multiple (of multiplicity 2), and by a small additional change of other parameters it can be split into two simple but complex roots. This is just in this way two virtual states collapse producing a resonance; a subject which deserves a special discussion (see the following section). Pairs of virtual states which may collapse are shown in figure 2 by squares and circles.

It is not difficult to convince oneself that for any given β_{jl} the situation when all the zeros of the Jost-matrix determinant are purely imaginary may be realized by a proper choice of α_j . To see that let us consider the asymptotic behavior of λ_j^{σ} for $|\bar{k}_1| \rightarrow \infty$, when all the off-diagonal entries of B become negligibly small,

$$\lambda_1^{\sigma} \simeq \bar{k}_1 + \alpha_1, \tag{30}$$

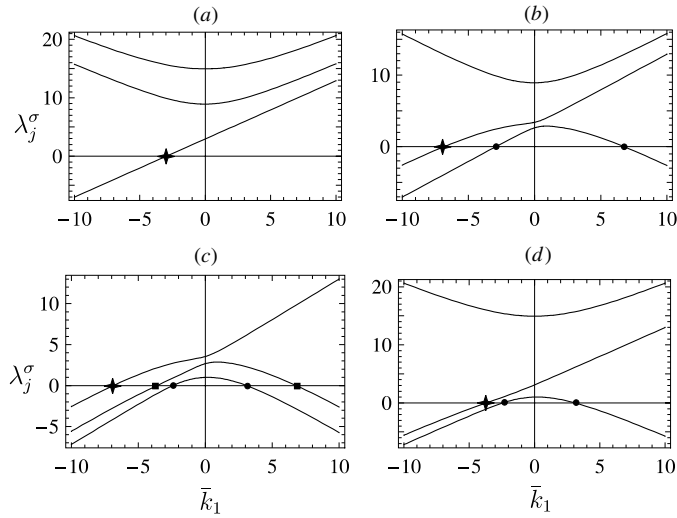


Figure 2. Typical behavior of the eigenvalues $\lambda_j^\sigma(\bar{k}_1)$, $N = 3$, is shown. Each plane corresponds to a particular choice of string σ : (a) $\sigma = (+++)$, (b) $\sigma = (++-)$, (c) $\sigma = (+--)$ and (d) $\sigma = (+-+)$. Stars, squares and circles correspond to the virtual states. Virtual states are denoted by the identical symbol if they belong to the same eigenvalue $\lambda_j^\sigma(\bar{k}_1)$. The corresponding parameters are $\alpha_1 = 3, \alpha_2 = 5, \alpha_3 = 9, \beta_{12} = 0.5, \beta_{13} = 0.4, \beta_{23} = 0.2, \Delta_2 = 15$ and $\Delta_3 = 35$.

$$\lambda_j^\sigma \simeq \sigma_j \sqrt{\bar{k}_1^2 + \Delta_j} + \alpha_j = \sigma_j \left(|\bar{k}_1| + \frac{\Delta_j}{2\bar{k}_1} + \dots \right) + \alpha_j, \quad (31)$$

$$|\bar{k}_1| \rightarrow \infty. \quad (32)$$

The numbers $n_+(\sigma)$ and $n_-(\sigma)$ determine the corresponding numbers of increasing and decreasing eigenvalues at positive infinity. The eigenvalue λ_1^σ increases both at negative and positive infinity. Now if we choose all α_j sufficiently large in absolute values and negative, we can always guarantee the location of a root of the equation $\lambda_1^\sigma(\bar{k}_1) = 0$ near the point $\bar{k}_1 = \alpha_1$ and at the same time the location of two roots of the equation $\lambda_j^\sigma(\bar{k}_1) = 0$ with corresponding $\sigma_j = +$ near the points $\bar{k}_1 = \pm\alpha_j$. Thus, for each σ we can obtain $2n_+(\sigma) + 1$ zeros. The total number, n_v , of these zeros may be calculated by formulae (27) and (28)

$$n_v = \sum_{\sigma} [2n_+(\sigma) + 1] = N2^{N-1}, \quad (33)$$

which coincides with the total number of all the possible roots of system (14), (15) and is just the maximal possible number of virtual states. Hence, in this case all the roots are purely imaginary. In the following section, we consider the case when some of the zeros may merge, become complex and produce resonances.

5. The number of resonances

For simplicity, independently of whether or not it can be seen in a scattering we call any pair of complex zeros $k = \pm k_r + ik_i$ of the Jost-matrix determinant a resonance keeping in mind that to be really visible in a scattering the resonance behavior of the corresponding cross section should be narrow and sharp enough.

Conservation of the number of zeros of an n th-order algebraic equation under a variation of parameters included in its coefficients, which keeps unchanged its order (in our case this is equation (19) obtained from system (14), (15)) applied to our case leads to the following relation $n_b + n_v + 2n_r = N2^{N-1}$, where n_b, n_v and n_r are the number of bound states, virtual states and resonances, respectively. The aim of this section is to establish the maximal number of possible resonances accepted by a non-conservative SUSY partner of the vanishing potential.

Evidently, the maximal number of resonances corresponds to the minimal number of bound n_b and virtual n_v states. These numbers would both become zero if none of the B matrix eigenvalues intersected the \bar{k}_1 axis. But as was noted in the previous section, there always exists an eigenvalue λ_1^σ with the asymptotic behavior given in (30), i.e. ranging from $-\infty$ to $+\infty$ and, hence, it intersects the \bar{k}_1 axis always and for all possible values of σ . We thus see that the minimal number of real zeros that all eigenvalues may take is achieved if all the eigenvalues $\lambda_j^\sigma(\bar{k}_1)$, $j > 1$, are nodeless and curves $\lambda_1^\sigma(\bar{k}_1)$ intersect the \bar{k}_1 axis only once for every given sign combination σ . To realize this case, we should choose the parameters included in U_0 in such a way that the global minimum $\lambda_{j,\min}^\sigma$ of every eigenvalue $\lambda_j^\sigma(\bar{k}_1)$ with $\sigma_j = +$ (they tend to $+\infty$ when $|\bar{k}_1| \rightarrow \infty$) be positive $\lambda_{j,\min}^\sigma > 0$ and, respectively, global maximum $\lambda_{j,\max}^\sigma$ of every eigenvalue $\lambda_j^\sigma(\bar{k}_1)$ with $\sigma_j = -$ (they tend to $-\infty$ when $|\bar{k}_1| \rightarrow \infty$) be negative $\lambda_{j,\max}^\sigma < 0$. Under these conditions only the eigenvalues $\lambda_1^\sigma(\bar{k}_1)$ have zeros. The possibility that these eigenvalues have only one zero can always be realized. This can be demonstrated for small enough values of β_{ij} (so-called weak coupling approximation, see the following section) which in the limit $\beta_{ij} = 0$ for all i, j gives a very simple behavior of the eigenvalues. For instance, for $\Delta_{j+1} - \Delta_j$ large enough and $\min_j(\sqrt{\Delta_j} + \alpha_j) > \max_j(-\sqrt{\Delta_j} + \alpha_j)$, the straight line $\lambda_1^\sigma(\bar{k}_1)$ never intersects with the hyperbolas $\lambda_j^\sigma(\bar{k}_1)$ so that small perturbations coming from small nonzero β_{ij} -values (in a physical terminology these perturbations shift the zero-width resonances from the real-energy axis to the complex plane) do not change the monotonous behavior of $\lambda_1^\sigma(\bar{k}_1)$ and, hence, do not bring additional roots to the equation $\lambda_1^\sigma(\bar{k}_1) = 0$.

Thus, we see that the minimal value of virtual states with the absence of bound states is equal to all possible sign combinations of σ which is $n_{\text{vmin}} = \sum_\sigma 1 = 2^{N-1}$. Hence, the maximal possible number of resonances is obtained by subtracting this number from the total number of solutions, i.e.

$$2n_{r,\text{max}} = N2^{N-1} - 2^{N-1} = (N - 1)2^{N-1}. \tag{34}$$

6. Weak coupling approximation

For the number of channels $N > 2$ there is no way to get analytical solutions of system (14), (15), but if the coupling parameters β_{ij} are small enough assuming analyticity of the roots of the Jost-matrix determinant as functions of β_{ij} a perturbation technique may be developed. In this section, we demonstrate this possibility by obtaining first-order corrections to unperturbed values of the roots of the Jost-matrix determinant corresponding to $\beta_{ij} = 0$.

For the zero coupling, the matrix U_0 becomes diagonal $U_0 = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_N)$ and system (11), (3) reduces to

$$(\alpha_1 - ik_{0,1})(\alpha_2 - ik_{0,2}) \cdots (\alpha_N - ik_{0,N}) = 0, \tag{35}$$

$$k_{0,j}^2 - k_{0,1}^2 + \Delta_j = 0, \quad j = 2, \dots, N, \tag{36}$$

where the additional subscript 0 corresponds to the uncoupled case. Its solutions have the form

$$\begin{aligned}
 k_{0,1}^{(1,\sigma)} &= -i\alpha_1, & k_{0,m}^{(1,\sigma)} &= \sigma_m \sqrt{-\alpha_1^2 - \Delta_m}, & m \neq 1, \\
 k_{0,2}^{(2,\sigma)} &= -i\alpha_2, & k_{0,m}^{(2,\sigma)} &= \sigma_m \sqrt{-\alpha_2^2 + \Delta_2 - \Delta_m}, & m \neq 2, \\
 &\dots & & & \\
 k_{0,N}^{(N,\sigma)} &= -i\alpha_N, & k_{0,m}^{(N,\sigma)} &= \sigma_m \sqrt{-\alpha_N^2 + \Delta_N - \Delta_m}, & m \neq N,
 \end{aligned} \tag{37}$$

where $m = 1, \dots, N$. Let us explicitly indicate the meaning of subscripts and superscripts in (37): the second subscript m in $k_{0,m}^{(j,\sigma)}$ corresponds to the channel, the first superscript j indicates a row number in (37) and σ indicates the one of all 2^{N-1} combinations of signs. Thus, we see once again that the total number of solutions of the system is $N2^{N-1}$ and it does not depend on whether or not the coupling is absent. Note that every energy level $E_j = -\alpha_j^2 + \Delta_j$ corresponding to a row in (37) is 2^{N-1} fold degenerate. Below we show that under a small coupling every degenerate level E_j splits by 2^{N-1} sub-levels and we will find approximate values of the splitting. But the unperturbed j th momentum corresponding to this level simply equals $k_{0,j}^{(j,\sigma)} = -i\alpha_j$. Therefore, instead of our previous convention to express all quantities in terms of k_1 , it is convenient here to express corrections to the j th momentum produced by a perturbation in terms of the unperturbed j th momentum $\bar{k}_{0,j}^{(j,\sigma)}$. This is always possible due to the fact that all momenta have equal rights. But now we have to change our sign conventions introduced in section 4 where the first momentum \bar{k}_1 entered in the string σ always with the positive sign ($\sigma_1 = +$). Now we have j th momentum $\bar{k}_j \in (-\infty, \infty)$ and $\sigma_j = +$ in string σ .

From (37), we learn that no coupling implies no finite-width resonances, but as we discuss below the zeros lying above the first threshold may be associated with zero-width resonances which acquire a nonzero width under a small coupling.

From the first row of (37), we conclude that the corresponding 2^{N-1} zeros with $E_1 = -\alpha_1^2$ are always below the first threshold (bound or virtual states). The energy $E_n = -\alpha_n^2 + \Delta_n, n = 2, \dots, N$, may be positive with respect to the first threshold and just these $(N-1)2^{N-1}$ zeros are associated with the zero-width resonances. According to our convention, a resonance corresponds to a pair of complex zeros. Here we can easily compute the number of the zero-width resonances, n_{zwr} , which is $n_{zwr} = (N-1)2^{N-2}$ which agrees with the maximal number of possible resonances obtained in the previous section.

The unperturbed B^σ matrix we denote by B_0^σ is diagonal,

$$B_0^\sigma = \text{diag}(\alpha_1 + \sigma_1 \bar{k}_1, \alpha_2 + \sigma_2 \bar{k}_2, \dots, \alpha_N + \sigma_N \bar{k}_N), \tag{38}$$

and its eigenvalues $\lambda_{0,j}^\sigma$ coincide with its diagonal entries,

$$\lambda_{0,j}^\sigma(\bar{k}_j) = \alpha_j + \bar{k}_j, \quad \lambda_{0,l}^\sigma(\bar{k}_j) = \alpha_l + \sigma_l \sqrt{\bar{k}_j^2 + \Delta_l - \Delta_j}, \tag{39}$$

$$l = 1, \dots, N, \quad l \neq j. \tag{40}$$

For simplicity, we assume all the coupling parameters β_{ij} proportional to the same small parameter β so that the perturbed B^σ matrix reads

$$B^\sigma = B_0^\sigma + \beta \mathcal{B}, \quad \mathcal{B} = \|b_{jl}\|, \quad b_{jj} = 0, \quad j = 1, \dots, N. \tag{41}$$

Now, as it was mentioned above assuming analyticity of the eigenvalues of this matrix as functions of β we can develop them in a Taylor series with respect to β ,

$$\tilde{\lambda}_j^\sigma = \lambda_{0,j}^\sigma + \lambda_{1,j}^\sigma + \lambda_{2,j}^\sigma + \dots, \tag{42}$$

where the first subscript number is just the power of β . First, we note that the perturbation \mathcal{B} has zero diagonal entries which results in $\lambda_{1,j}^\sigma = 0$. To get the second-order correction, we are using the usual Rayleigh–Schrödinger perturbation approach which leads to

$$\lambda_{2,j}^\sigma(\bar{k}_j) = \beta^2 \sum_{l=1, l \neq j}^N \frac{b_{jl}^2}{\lambda_{0,j}^\sigma(\bar{k}_j) - \lambda_{0,l}^\sigma(\bar{k}_j)}. \tag{43}$$

In what follows, we also assume that we can neglect the higher-order corrections to the eigenvalues.

Actually, our aim is to find corrections to the unperturbed degenerate j th Jost-matrix determinant zero given in (37). Assuming the Taylor series expansions for this root over the small parameter β indicating it now explicitly

$$\bar{k}_j = \bar{k}_{0,j}^{(j,\sigma)} + \beta c_1 + \beta^2 c_2 + \dots \tag{44}$$

we find the coefficients c_1 and c_2 from the equation

$$\tilde{\lambda}_j^\sigma(\bar{k}_j) = \lambda_{0,j}^\sigma(\bar{k}_j) + \lambda_{2,j}^\sigma(\bar{k}_j) = 0. \tag{45}$$

For that we develop $\tilde{\lambda}_j^\sigma(\bar{k}_j)$ in a Taylor series in β parameter considering its β dependence as given through \bar{k}_j and (44). Equation (43) contains the factor β^2 , therefore in its denominator we simply put $\bar{k}_{0,j}^{j,\sigma}$ instead of \bar{k}_j . The \bar{k}_j -dependence of the term $\lambda_{0,j}^\sigma(\bar{k}_j)$ is given by (39) and its β -dependence is obtained via (44). Thus, the left-hand side of equation (45) is presented as a series over the powers of β where every coefficient should vanish. This leads to $c_1 = 0$ and

$$c_2 = \sum_{l=1, l \neq j}^N \frac{b_{jl}^2}{\alpha_l + \sigma_l \sqrt{\alpha_j^2 + \Delta_l} - \Delta_j}. \tag{46}$$

Finally, up to the second order in β , we obtain the roots of system (37):

$$\begin{aligned} k_1^{(1,\sigma)} &= -i\alpha_1 + i \sum_{l=2}^N \frac{\beta^2 b_{1l}^2}{\alpha_l + \sigma_l \sqrt{\alpha_1^2 + \Delta_l}}, & k_m^{(1,\sigma)} &= \sigma_m \sqrt{(k_1^{(1,\sigma)})^2 - \Delta_m}, \\ k_2^{(2,\sigma)} &= -i\alpha_2 + i \sum_{l=1, l \neq 2}^N \frac{\beta^2 b_{2l}^2}{\alpha_l + \sigma_l \sqrt{\alpha_2^2 + \Delta_l} - \Delta_2}, & k_m^{(2,\sigma)} &= \sigma_m \sqrt{(k_2^{(2,\sigma)})^2 + \Delta_2 - \Delta_m}, \\ &\dots & & \\ k_N^{(N,\sigma)} &= -i\alpha_N + i \sum_{l=1}^{N-1} \frac{\beta^2 b_{Nl}^2}{\alpha_l + \sigma_l \sqrt{\alpha_N^2 + \Delta_l} - \Delta_N}, & k_m^{(N,\sigma)} &= \sigma_m \sqrt{(k_N^{(N,\sigma)})^2 + \Delta_N - \Delta_m}. \end{aligned} \tag{47}$$

Here each row is obtained by applying equations (43), (44), (45) and (46) for $j = 1, \dots, N$, respectively, and $m = 1, \dots, N, m \neq j$ for each j . The square roots in the last column of (47) should be expanded in the Taylor series up to β^2 .

From here it is easily seen that, when $\alpha_m^2 < \Delta_m$, purely imaginary unperturbed zeros $k_m = -i\alpha_m$ move from the axes to the complex plane due to the real part of corrections. For instance, for k_2 , the real part reads $\pm \beta^2 \sqrt{\Delta_2 - \alpha_2^2} / (\alpha_1^2 - \alpha_2^2 + \Delta_2)$. We thus confirmed the previous statement that zero-width resonances acquire nonzero widths.

7. Zeros of the Jost-matrix determinant for $N = 2$

The particular case of two coupled channels is important both from practical and theoretical point of view. Let us recall the following inequalities for the number of bound/virtual states

and resonances obtained in sections 3, 4 and 5: $0 \leq n_b \leq 2$, $0 \leq n_r \leq 1$ and $0 \leq n_v \leq 4$. The same inequalities are obtained for $N = 2$ in [10, 11] from another approach. The two-channel problem is the only one where one is able to get analytic expressions for the Jost-determinant roots, i.e. to solve the direct problem consisting in finding the positions of the bound/virtual states and resonances. This possibility is based on the fact that the roots of the algebraic equation of fourth order, $(N2^{N-1})|_{N=2} = 4$, may be expressed in radicals. Thus we obtain zeros as the functions of parameters defining the potential. One may be interested in solving the inverse problem: to express parameters of the potential from the knowledge about positions of the zeros of the Jost-matrix determinant. In principle, one may try to inverse radicals, but we propose a more elegant way below.

To simplify the notations, we choose in this case $\Delta_2 \equiv \Delta \neq 0$. The potential, which is known as the Cox potential [8], depends on three parameters appearing in the matrix

$$U_0 = \begin{pmatrix} \alpha_1 & \beta \\ \beta & \alpha_2 \end{pmatrix}, \tag{48}$$

and on the factorization energy \mathcal{E} which is upper bounded. The Jost-matrix determinant reads

$$f(k_1, k_2) \equiv \det F(k_1, k_2) = \frac{(k_1 + i\alpha_1)(k_2 + i\alpha_2) + \beta^2}{(k_1 + i\kappa_1)(k_2 + i\kappa_2)}. \tag{49}$$

The system of equations (14), (15) in this case takes the simplest form

$$k_1^2 - k_2^2 = \Delta, \tag{50}$$

$$(k_1 + i\alpha_1)(k_2 + i\alpha_2) + \beta^2 = 0 \tag{51}$$

and can be reduced to a fourth-order algebraic equation with respect to k_1

$$k_1^4 + ia_1k_1^3 + a_2k_1^2 + ia_3k_1 + a_4 = 0. \tag{52}$$

The coefficients $a_i, i = 1, \dots, 4$, are given explicitly in [10, equations (33a)–(33d)]. The momentum k_2 can be found from

$$k_2(ik_1 - \alpha_1) = \alpha_2(k_1 + i\alpha_1) - i\beta^2, \tag{53}$$

which is a direct implication of (51). Equations (52) and (53) are the particular case of system (19), (18), . . . , (16) for $N = 2$ in accordance with our general discussion in section 2.

Let us assume we have found two of the roots of system (50), (51) we denote $(k_1^{(1)}, k_2^{(1)})$ and $(k_1^{(2)}, k_2^{(2)})$, which clearly are the functions of parameters α_1 and α_2 . Their dependence on the parameters β and Δ is not important for the moment, since both β and Δ assumed to be fixed. Being put back to (51), the equation reduces twice to identity for any values of α_1 and α_2 , which we write as

$$(k_1^{(1)} + i\alpha_1)(k_2^{(1)} + i\alpha_2) + \beta^2 = 0, \tag{54}$$

$$(k_1^{(2)} + i\alpha_1)(k_2^{(2)} + i\alpha_2) + \beta^2 = 0. \tag{55}$$

The reason why we replaced the identity sign by the equality sign is that these equations may be considered as an implicitly written inverted dependence of $\alpha_{1,2}$ on the set of parameters $k_{1,2}^{(1,2)}$. We may thus fix arbitrary values for $k_{1,2}^{(1,2)}$ and find from (54), (55) α_1 and α_2 in terms of $k_{1,2}^{(1,2)}$ which is a much easier task than finding an explicit dependence of $k_{1,2}^{(1,2)}$ on α_1 and α_2 . For that we have to solve, e.g. for α_1 , the following second-order equation:

$$\alpha_1^2 - \alpha_1 i(k_1^{(1)} + k_1^{(2)}) - k_1^{(1)}k_1^{(2)} + \beta^2 \frac{R_1}{R_2} = 0, \tag{56}$$

with $R_1 = k_1^{(2)} - k_1^{(1)}$ and $R_2 = k_2^{(2)} - k_2^{(1)}$ which easily follows from (54) and (55). From here we find

$$\alpha_1 = \frac{1}{2}[i(k_1^{(1)} + k_1^{(2)}) \pm \sqrt{-R_1^2 - 4\beta^2 R_1/R_2}], \quad (57)$$

$$\alpha_2 = \frac{1}{2}[i(k_2^{(1)} + k_2^{(2)}) \mp \sqrt{-R_2^2 - 4\beta^2 R_2/R_1}]. \quad (58)$$

The upper (lower) sign in (58) corresponds to the upper (lower) sign in (57). The values of $k_1^{(1,2)}$ and $k_2^{(1,2)}$ should be chosen so as to warranty the reality of parameters $\alpha_{1,2}$.

Once the two roots are fixed, (52) reduces to a second-order algebraic equation $\mathcal{Q}_2(k_1) = 0$ for the two other roots $k_1^{(3)}$ and $k_1^{(4)}$, thus providing an implicit but rather simple mapping between the roots of system (50), (51) and the set of parameters $(\alpha_1, \alpha_2, \beta)$. The polynomial $\mathcal{Q}_2(k_1)$ is the ratio of the polynomial appearing in (52) and $\mathcal{P}_2(k_1) = k_1^2 - k_1(k_1^{(2)} + k_1^{(1)}) + k_1^{(2)}k_1^{(1)}$, i.e.,

$$k_1^4 + ia_1k_1^3 + a_2k_1^2 + ia_3k_1 + a_4 = \mathcal{P}_2(k_1)\mathcal{Q}_2(k_1).$$

From here we find, with the explicit expression for coefficients $a_i, i = 1, \dots, 4$ [10],

$$\mathcal{Q}_2(k_1) = (k_1 + i\alpha_1)^2 + k_1(k_1^{(2)} + k_1^{(1)}) + (2i\alpha_1 + k_1^{(2)} + k_1^{(1)})(k_1^{(2)} + k_1^{(1)}) + \alpha_2^2 - \Delta - k_1^{(1)}k_1^{(2)}$$

and, hence,

$$k_1^{(3)} = \frac{1}{2}[\mp i\sqrt{-R_1^2 - 4\beta^2 R_1/R_2} + \sqrt{D_1}], \quad (59)$$

$$k_1^{(4)} = \frac{1}{2}[\mp i\sqrt{-R_1^2 - 4\beta^2 R_1/R_2} - \sqrt{D_1}], \quad (60)$$

where $D_1 = R_1^2 + 4\beta^2 \frac{R_2}{R_1} + 4k_1^{(2)}k_1^{(1)}$. The sign before the first square root in (59) and (60) should be chosen in accordance with the signs in (57) and (58).

To find $k_2^{(3,4)}$ we do not need to solve any equation. We simply note that the equation $\det F(k_1, k_2) = 0$ is invariant under the transformation $k_1 \leftrightarrow k_2, \alpha_1 \leftrightarrow \alpha_2, \Delta \leftrightarrow -\Delta$. This means that being transformed according to these rules equations (59) and (60) give us the k_2 values:

$$k_2^{(3)} = \frac{1}{2}[\mp i\sqrt{-R_2^2 - 4\beta^2 R_2/R_1} - \sqrt{D_2}], \quad (61)$$

$$k_2^{(4)} = \frac{1}{2}[\mp i\sqrt{-R_2^2 - 4\beta^2 R_2/R_1} + \sqrt{D_2}], \quad (62)$$

where $D_2 = R_2^2 + 4\beta^2 \frac{R_1}{R_2} + 4k_2^{(2)}k_2^{(1)}$.

Two initial zeros $(k_1^{(1)}, k_2^{(1)})$, $(k_1^{(2)}, k_2^{(2)})$ and the threshold difference Δ are assumed to be known from the experiment. For instance, these zeros may correspond to a visible Feshbach resonance or two bound states. The possible cases for initial data are summarized in table 1. The first row of table 1 corresponds to the case where the position of the resonance is known (see section 7.1). The second row corresponds to the case where the positions of both the resonance and one bound state are known, which allows one to fix the maximal number of parameters. The third row corresponds to the case where the positions of two bound states are known (see section 7.2). The last row corresponds to the special case when only one zero may be fixed from experimental data. The free parameters in table 1 allow either for isospectral deformations of the potential or for fits of additional experimental data as, e.g., scattering lengths (see, e.g. [10, 11]). The restriction on the factorization energy is deduced from the regularity condition of potential (10). The restriction on the coupling parameter β is explained

Table 1. Possible mappings between some experimental data and the Cox potential parameters.

Experimental data	Fixed parameters	Free parameters	Restrictions
Δ, E_r, E_i	α_1, α_2	κ_1, β	$\beta \geq \sqrt{-k_r p_r}$
$\Delta, E_b = -\lambda_b^2, E_r, E_i$	$\alpha_1, \alpha_2, \beta$	κ_1	$\kappa_1 > \lambda_b$
$\Delta, E_{1,2} = -\lambda_{1,2}^2$	α_1, α_2	κ_1, β	$\kappa_1 > \lambda_2 > \lambda_1$
$\Delta, E_b = -\lambda_b^2$	α_2	$\kappa_1, \beta, \alpha_1$	$\kappa_1 > \lambda_b$

below (see (67)). Let us now consider the examples corresponding to the first three rows of table 1.

7.1. One resonance

A resonance corresponds to a pair of complex roots $k_1^{(1)}$ and $k_1^{(2)}$ of the Jost-matrix determinant such that $ik_1^{(1)}$ and $ik_1^{(2)}$ are mutually complex conjugates. Therefore, we assume equations (50) and (51) to have two complex roots. Let us define their first-channel components as

$$k_1^{(1)} = k_r + ik_i, \quad k_1^{(2)} = -k_r + ik_i, \quad k_i \in \mathbb{R}, \quad k_r \in \mathbb{R}, \quad k_r > 0, \quad (63)$$

and write the corresponding energies, $(k_1^{(1,2)})^2$, as $E_r \pm iE_i$, where we also assume $E_i > 0$ (which means that the upper sign corresponds to $k_1^{(1)}$ or $k_1^{(2)}$, depending on the sign of k_i). We would like to choose as parameters the threshold difference Δ , as well as the real and imaginary parts of the resonance complex energy, E_r, E_i . As exemplified below, these can correspond to physical parameters of a visible resonance in some (but not all) cases. In terms of these parameters, k_r and k_i are expressed as

$$k_r = \frac{E_i}{\sqrt{2}} [\sqrt{E_r^2 + E_i^2} - E_r]^{-1/2}, \quad k_i = \pm \frac{1}{\sqrt{2}} [\sqrt{E_r^2 + E_i^2} - E_r]^{1/2}. \quad (64)$$

In the second channel, the roots

$$k_2^{(1)} = p_r + ip_i, \quad k_2^{(2)} = -p_r + ip_i$$

can be found from the threshold condition yielding

$$p_r = -\frac{1}{\sqrt{2}} [\sqrt{E_i^2 + (E_r - \Delta)^2} + E_r - \Delta]^{1/2}, \quad (65)$$

$$p_i = \mp \frac{E_i}{\sqrt{2}} [\sqrt{E_i^2 + (E_r - \Delta)^2} + E_r - \Delta]^{-1/2}. \quad (66)$$

The upper (lower) sign in (64) corresponds to the upper (lower) sign in (66), which means that, for a given zero, the signs of k_i and p_i are opposite. Moreover, equations (64) and (65) show that, for a given zero, the signs of k_r and p_r are also opposite. This implies that, for the Cox potential, the complex resonance zeros (or scattering-matrix poles) are always in opposite quadrants in the complex k_1 and k_2 planes. This has important consequences for physical applications: for a resonance to be visible, one of the corresponding zeros has to lie close to the physical positive-energy region, i.e., close to the real-positive k_1 axis and close to the region made of the real-positive k_2 axis and of the positive-imaginary k_2 interval: $[0, i\sqrt{\Delta}]$. Consequently, the only possibility for a visible resonance with the Cox potential is that of a Feshbach resonance, only visible in the channel with lowest threshold, with an energy lying

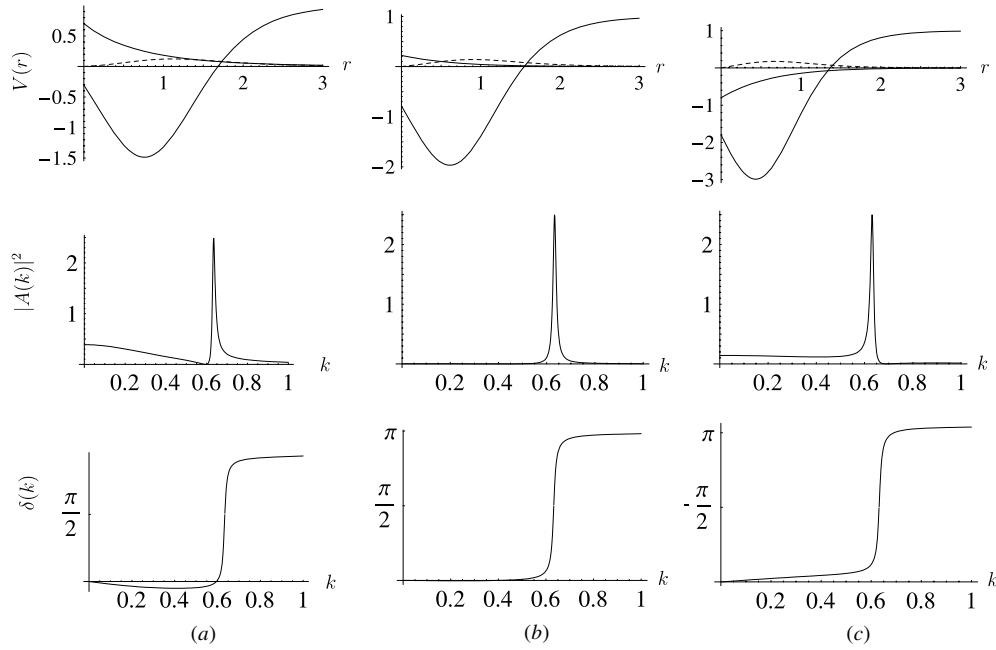


Figure 3. The Cox potential without bound state and with one visible resonance of energy $E_r = 0.4$ and width $\Gamma = 0.02$, for $\Delta = 1$ and $\beta = 0.1$ (first row, solid lines for V_{11} and $V_{22} + \Delta$, dashed line for V_{12}), with the corresponding partial cross section (second row) and phase shifts (third row) for (a) $\kappa_1 = 0.5$, (b) $\kappa_1 = 0.7$ and (c) $\kappa_1 = 1$.

below the threshold Δ . At higher resonance energies, the corresponding zero is either close to the k_1 -plane physical region (and far from the k_2 -plane one) or close to the k_2 -plane physical region (and far from the k_1 -plane one); it cannot be close to both physical regions at the same time, hence it cannot have a visible impact on the coupled scattering matrix. Here, we illustrate the case of a visible resonance, which is the most interesting from the physical point of view. It corresponds to the lower signs in (64) and (66), with a resonance energy E_r such that $0 < E_r < \Delta$, and a resonance width $\Gamma = 2E_i$ such that $E_i < E_r$.

Note that for nonzero values of the parameters k_r and p_r (which have opposite signs), the coupling parameter β cannot be infinitesimal: because α_1 and α_2 have to be real, β is restricted to satisfy the inequality

$$\beta \geq \sqrt{-k_r p_r}. \tag{67}$$

To get a potential with one bound state at energy $-\lambda_b^2$, we choose the lower signs in (57), (58). We then get for $k_1^{(3)}(\beta)$ an expression similar to (59), (60), from which the value of β can be found by solving the bi-squared equation $k_1^{(3)}(\beta) = i\lambda_b$.

Let us now choose explicit parameters. First, we put $\Delta = 1$. To get a visible resonance, we put $E_r = 0.4$, $E_i = 0.01$ (which corresponds to a resonance width $\Gamma = 0.02$), and $\beta = 0.1$. Using (57), (63) and (64), one finds $\alpha_1 = 0.76938$ and $\alpha_2 = -0.766853$ (we choose the upper signs (57), (58)). The factorization energy, \mathcal{E} , is not constrained in this case: it just has to be negative. The Cox potential with one resonance and two virtual states $E_{v1} = -0.560473$, $E_{v2} = -0.599544$ is shown in the first row of figure 3.

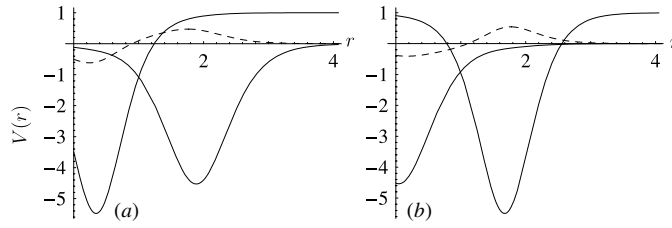


Figure 4. The Cox potential (solid lines for V_{11} and $V_{22} + \Delta$, dashed line for V_{12}) with two bound states at energies $E_1 = -0.01$ and $E_2 = -2.25$, for $\Delta = 1$, $\beta = 0.1$ and $\kappa_1 = 1.51$. The left (right) graphic corresponds to the upper (lower) signs in (57) and (58).

The diagonal elements of the potentials, V_{11} and $V_{22} + \Delta$, are plotted with solid lines, while V_{12} is plotted with dashed lines. The parameter κ_1 is responsible for the isospectral deformation of the potential which results in the behavior of the phase shifts. The second row of figure 3 shows the corresponding partial cross sections, where the resonance behavior is clearly seen, as well as the evolution of the low-energy cross section, which is related to the scattering length. The last row of figure 3 shows the corresponding phase shifts for the open channel, where a typical Breit–Wigner behavior (see, e.g. [1]) is seen for the resonance, as well as the evolution of the zero-energy phase-shift slope, which is also related to the scattering length.

7.2. Two bound states

Let us now construct a Cox potential with two bound states, and hence no resonance [10]. We choose $k_1^{(1)} = 0.1i$ and $k_1^{(2)} = 1.5i$ for these bound states and, as in the previous example, we put $\Delta = 1$ and $\beta = 0.1$. We thus have $k_2^{(1)} = \sqrt{1.01}i$ and $k_2^{(2)} = \sqrt{3.25}i$, which defines R_2 in (57), (58). Choosing the upper signs in these equations, we find $\alpha_1 = -0.112649$ and $\alpha_2 = -1.79557$, while for the lower signs, we get $\alpha_1 = -1.48735$ and $\alpha_2 = -1.0122$. The corresponding Cox potentials are shown in figure 4.

8. Conclusion

A careful study of spectral properties of non-conservative multichannel SUSY partners of the zero potential is given. Our treatment is based on the analysis of the Jost-matrix determinant zeros. Generalizing our previous results for the two-channel case [10, 11], we have shown that the zeros of the Jost-matrix determinant are the roots of an $N2^{N-1}$ th-order algebraic equation. The number of bound states n_b is restricted by the number of channels, $0 \leq n_b \leq N$. The upper bound for the number of resonances is $(N - 1)2^{N-2}$. The generalization is based on the analysis of the behavior of the Jost-matrix eigenvalues.

In general, an algebraic equation of an order higher than 4 has no solutions in radicals. As a consequence, there are no exact analytic solutions of a spectral problem for a non-conservatively SUSY-transformed Hamiltonian with $N > 2$. Therefore, the problem of finding the approximate solutions appears to be actual. Based on the usual Rayleigh–Schrödinger perturbation theory for the eigenvalues of the Jost matrix, we develop an approximate method for finding the zeros of the Jost-matrix determinant in the case of a weak coupling between channels.

An analytical study of the Jost-determinant zeros is carried out for the two-channel case which implies an algebraic equation of the fourth order. A suitable factorization of the fourth-order polynomial allows us to develop a procedure which solves the inverse spectral problem for this case. The effectiveness of the procedure is illustrated by two examples: a potential with one resonance and a potential with two bound states.

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